

Bernstein modal basis: application to the spectral Petrov-Galerkin method for fractional partial differential equations

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Abstract In the spectral Petrov-Galerkin methods, the trial and test functions are required to satisfy particular boundary conditions. By a suitable linear combination of orthogonal polynomials, a basis, that is called the modal basis, is obtained. In this paper, we extend this idea to the non-orthogonal dual Bernstein polynomials. A compact general formula is derived for the modal basis functions based on dual Bernstein polynomials. Then, we present a Bernstein-spectral Petrov-Galerkin method for a class of time fractional partial differential equations with Caputo derivative. It is shown that the method leads to banded sparse linear systems for problems with constant coefficients. Some numerical examples are provided to show the efficiency and the spectral accuracy of the method.

Keywords Bernstein polynomials · Petrov-Galerkin · Dual Bernstein polynomials · Fractional partial differential equations · Modal basis

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1 Introduction

Due to the interesting features like shape preserving [3], optimal stability [7], etc., Bernstein polynomials are commonly used in computer aided geometric design (CAGD) for approximating curves and surfaces and designing computer fonts [6]. They have been applied in popular programs such as Adobe's Illustrator, Flash and Postscript in the form of Bézier curves [15].

Bernstein polynomials have also been implemented for solving differential, integro-differential and fractional differential equations [1, 5, 13, 17]. However, they are not orthogonal, leading to dense linear systems. The dual Bernstein polynomials (DBP) were explicitly presented by Juttler in 1998 [14]. To the best of our knowledge, they have been discussed only from CAGD point of view [16, 19].

The main purpose of this work is to derive a new polynomial basis by using the DBPs that can be used with the Petrov-Galerkin formulation for the boundary value problems of any order. It can also be used for solving fractional differential equations. We present a Bernstein-spectral Petrov-Galerkin method for a class of time fractional partial differential equations. It is shown that the method leads to

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banded matrices for problems with constant coefficients, saving in the computational costs for a desired accuracy. The spectral methods utilize high order basis functions, typically the orthogonal polynomials that are the solutions of the Sturm-Liouville equation and they are known to have spectral accuracy, i.e., having convergence speed faster than the methods with fixed polynomial rate of convergence like the finite element and finite difference methods, for problems with smooth solution.

Fractional PDEs play a key role in modeling some physical phenomena such as particle transport process in anomalous diffusion which has applications in semiconductors, finance, electrochemistry, etc. [2, 4, 8, 10]. The Caputo temporal fractional derivative of $u(x, t)$ is defined as

$$\partial_t^\alpha u(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^\alpha} \frac{\partial u(x, s)}{\partial s} ds, \quad 0 < \alpha < 1. \quad (1.1)$$

The paper is organized as follows. Section 2 gives some preliminaries of Bernstein and DBPs. Some new aspects of these polynomials and an interesting formula for the modal basis functions are presented in Section 3. In Section 4, a Bernstein-spectral Petrov-Galerkin method is developed for a class of time fractional differential equations. The efficiency and spectral accuracy of the method are illustrated via numerical examples in Section 5.

2 Bernstein polynomials and DBPs

Bernstein basis polynomials of degree N over the unit interval $I = [0, 1]$ are defined by

$$\phi_i := B_{i,N}(x) = \binom{N}{i} x^i (1-x)^{N-i}, \quad 0 \leq i \leq N. \quad (2.1)$$

We adopt the convention $\phi_i(x) \equiv 0$ for $i < 0$ and $i > N$. The set $\{\phi_i(x) : 0 \leq i \leq N\}$ forms a basis for P_N , the space of polynomials with degree not exceeding N . It enjoys interesting properties facilitating the numerical implementation. They possess the end-point interpolation property [11]

$$\phi_i(0) = \delta_{i,0}, \quad \phi_i(1) = \delta_{i,N}, \quad (2.2)$$

$$\phi_i^{(p)}(0) = \frac{(-1)^{i+p} N!}{(N-p)!} \binom{p}{i}, \quad p \leq N, \quad (2.3)$$

$$\phi_i^{(p)}(1) = \frac{(-1)^{N-i} N!}{(N-p)!} \binom{p}{N-i}, \quad p \leq N, \quad (2.4)$$

for $0 \leq i \leq N$. Especially, for $n \in \mathbb{N}$, $n \leq N$, the polynomials $\phi_i(x)$, $\lfloor \frac{n}{2} \rfloor \leq i \leq N - \lfloor \frac{n+1}{2} \rfloor$, satisfy the end-point conditions

$$\phi_i^{(p)}(0) = \phi_i^{(p)}(1) = 0, \quad 0 \leq p \leq \lfloor \frac{n}{2} \rfloor - 1, \quad (2.5)$$

where $\lfloor \cdot \rfloor$ indicates the floor function. Moreover, when n is an odd integer, we get

$$\phi_i^{(\lfloor \frac{n}{2} \rfloor)}(1) = 0, \quad \lfloor \frac{n}{2} \rfloor \leq i \leq N - \lfloor \frac{n+1}{2} \rfloor. \quad (2.6)$$

We will use (2.5) and (2.6) to introduce a basis for the solution fractional partial differential equations.

The derivative of Bernstein polynomials satisfies a three-term recurrence formula [11]

$$\phi'_i(x) = (N - i + 1) \phi_{i-1}(x) - (N - 2i) \phi_i(x) - (i + 1) \phi_{i+1}(x), \quad 0 \leq i \leq N. \quad (2.7)$$

The dual Bernstein polynomials given by

$$\tilde{\psi}_i(x) = \sum_{j=0}^N c_{ij} \phi_j(x), \quad (2.8)$$

with the coefficients

$$c_{ij} = \frac{(-1)^{i+j}}{\binom{N}{i} \binom{N}{j}} \sum_{r=0}^{\min(i,j)} (2r+1) \binom{N+r+1}{N-i} \binom{N-r}{N-i} \binom{N+r+1}{N-j} \binom{N-r}{N-j},$$

provide the following biorthogonality system

$$(\phi_i, \tilde{\psi}_j) = \delta_{i,j}, \quad 0 \leq i, j \leq N. \quad (2.9)$$

with the standard L^2 inner product $(f, g) = \int_I f g dx$ [14]. The matrix $\mathbf{C} = [c_{i,j} : 0 \leq i, j \leq N]$ is bisymmetric, i.e., $c_{i,j} = c_{j,i} = c_{N-i, N-j}$. It is also seen that

$$\sum_{i=0}^N c_{i,j} = \sum_{j=0}^N c_{i,j} = N + 1, \quad 0 \leq i, j \leq N. \quad (2.10)$$

This can be proved by (2.8) and changing the order of the double summation.

3 Modal basis functions

Using a suitable linear combination of a known orthogonal basis, typically the Jacobi polynomial basis, one may form a basis for the spectral Petrov-Galerkin method (see e.g. [9, 20]). We extend this idea to the non-orthogonal dual Bernstein polynomials. In this section, a compact formula for the modal basis functions is presented for an arbitrary order boundary value problem (BVP).

Consider a BVP of order n with boundary conditions

$$\begin{aligned} v^{(i)}(0) = v^{(i)}(1) = 0, \quad 0 \leq i \leq \frac{n-2}{2}, & \quad \text{for } n \text{ even,} \\ v^{(i)}(0) = v^{(i)}(1) = 0, \quad 0 \leq i \leq \frac{n-3}{2}, \text{ and } v^{(\frac{n-1}{2})}(1) = 0, & \quad \text{for } n \text{ odd.} \end{aligned} \quad (3.1)$$

There is no loss of generality in assuming homogeneous conditions. We define the *trial space* from which we seek an approximate solution of the problem as $V_N^{0,n} = \{v \in P_N : v \text{ satisfies the conditions (3.1)}\}$. We also define the *test space* $W_N^{0,n}$, as the set of polynomials w in P_N such that

$$\begin{aligned} w^{(i)}(0) = w^{(i)}(1) = 0, \quad 0 \leq i \leq \frac{n-2}{2}, & \quad \text{for } n \text{ even,} \\ w^{(i)}(0) = w^{(i)}(1) = 0, \quad 0 \leq i \leq \frac{n-3}{2}, \text{ and } w^{(\frac{n-1}{2})}(0) = 0, & \quad \text{for } n \text{ odd.} \end{aligned} \quad (3.2)$$

A basis in $W_N^{0,n}$ is chosen to serve as the *test functions* in Petrov-Galerkin formulation of the problem. Note that $\dim V_N^{0,n} = \dim W_N^{0,n} = N - n + 1$. Also, $V_N^{0,n} = W_N^{0,n}$ when n is even.

From (2.5)-(2.6), it is seen that the set

$$\{\phi_i(x) : \lfloor \frac{n}{2} \rfloor \leq i \leq N - \lfloor \frac{n+1}{2} \rfloor\}, \quad (3.3)$$

forms a basis for $V_N^{0,n}$. Before presenting a basis for $W_N^{0,n}$, we provide the following results for DBPs.

Lemma 1 [12] Set $\alpha_{i,0} := -(-1)^i(N+1)\binom{N+1}{i+1} + N\delta_{i,0} + \delta_{i,1}$ for $0 \leq i \leq N$. Then,

$$\begin{aligned} \tilde{\psi}'_i(x) = & \alpha_{i,0}\tilde{\psi}_0(x) + (1 - \delta_{i,1})i\tilde{\psi}_{i-1}(x) + (1 - \delta_{i,0})(1 - \delta_{i,N})(N - 2i)\tilde{\psi}_i(x) \\ & - (1 - \delta_{i,N-1})(N - i)\tilde{\psi}_{i+1}(x) - \alpha_{N-i,0}\tilde{\psi}_N(x), \end{aligned} \quad (3.4)$$

where we set $\tilde{\psi}_i \equiv 0$ for $i < 0$ and $i > N$.

Proposition 1 The following statements hold for $0 \leq i \leq N$ and $x \in I = [0, 1]$:

$$\begin{aligned} (a) \quad & \tilde{\psi}_{N-i}(x) = \tilde{\psi}_i(1-x), \quad (b) \quad \sum_{j=0}^N \tilde{\psi}_j(x) = N+1, \\ (c) \quad & \int_0^1 \tilde{\psi}_i(x) dx = 1, \quad (d) \quad \tilde{\psi}_i^{(p)}(0) = \frac{(-1)^p N!}{(N-p)!} \sum_{r=0}^p (-1)^r c_{i,r} \binom{p}{r}. \end{aligned}$$

Proof The first statement follows from definition (2.8) and the similar relation $\phi_{N-i}(x) = \phi_i(1-x)$. From (2.8) and (2.10) we get the following that proves (b):

$$\sum_{i=0}^N \tilde{\psi}_i(x) = \sum_{i=0}^N \sum_{j=0}^N c_{i,j} \phi_j(x) = \sum_{j=0}^N \phi_j(x) \sum_{i=0}^N c_{i,j} = N+1.$$

The statement (c) follows from the fact $\int_0^1 \phi_i(x) dx = \frac{1}{N+1}$ and (2.10). (d) is derived by (2.3). \square

The following theorem gives a formula for the modal basis functions for the test space $W_N^{0,n}$.

Theorem 1 Let $n < N$. The following polynomials form a basis for $W_N^{0,n}$.

$$\psi_i(x) = \sum_{j=0}^n a_{i,j}^n \tilde{\psi}_{i+j}(x), \quad 0 \leq i \leq N-n, \quad (3.5)$$

$$a_{i,j}^n = \frac{\binom{n}{j}(i+j + [\frac{n+1}{2}])!(N-i-j + [\frac{n}{2}])!}{(i + [\frac{n+1}{2}])!(N-i + [\frac{n}{2}])!}, \quad 0 \leq j \leq n. \quad (3.6)$$

Proof The leading coefficient in (3.5) is $a_{i,0}^n = 1$, $\tilde{\psi}_i$'s are linearly independent and the number of ψ_i 's are equal to $\dim W_N^{0,n} = N-n+1$. It is thus sufficient to prove that $\psi_i \in W_N^{0,n}$ for $0 \leq i \leq N-n$. If n is even, it is

$$\psi_i^{(p)}(0) = \psi_i^{(p)}(1) = 0, \quad 0 \leq p \leq \frac{n-2}{2}.$$

To do this, using Proposition 1, we have

$$\begin{aligned} \psi_i^{(p)}(0) &= \sum_{j=0}^n a_{i,j}^n \tilde{\psi}_{i+j}^{(p)}(0) \\ &= \frac{(-1)^p N!}{(N-p)!} \sum_{j=0}^n a_{i,j}^n \sum_{r=0}^p (-1)^r c_{i+j,r} \binom{p}{r} \\ &= \frac{(-1)^p N!}{(N-p)!} \sum_{r=0}^p (-1)^r \binom{p}{r} \sum_{j=0}^n a_{i,j}^n c_{i+j,r}. \end{aligned}$$

With some manipulations, it is seen that the inner summation vanishes for $0 \leq r \leq p$, hence the proof is completed. The proof for odd n is done similarly. \square

For example, the modal basis functions (3.5) for $W_N^{0,2}$, $W_N^{0,3}$ and $W_N^{0,4}$ are written as

$$\psi_i = \tilde{\psi}_i + \frac{i+2}{N-i+1}(2\tilde{\psi}_{i+1} + \frac{i+3}{N-i}\tilde{\psi}_{i+2}), \quad 0 \leq i \leq N-2, \quad (3.7)$$

$$\psi_i = \tilde{\psi}_i + \frac{i+2}{N-i+2}(3\tilde{\psi}_{i+1} + \frac{i+3}{N-i+1}(3\tilde{\psi}_{i+2} + \frac{i+4}{N-i}\tilde{\psi}_{i+3})), \quad 0 \leq i \leq N-3, \quad (3.8)$$

$$\psi_i = \tilde{\psi}_i + \frac{i+3}{N-i+3}(4\tilde{\psi}_{i+1} + \frac{i+4}{N-i+1}(6\tilde{\psi}_{i+2} + \frac{i+5}{N-i}(4\tilde{\psi}_{i+3} + \frac{i+6}{N-i-1}\tilde{\psi}_{i+4}))), \quad 0 \leq i \leq N-4, \quad (3.9)$$

respectively. These are used for second, third and fourth order differential equations with conditions (3.1), respectively. As in the finite element method, the advantage of using such a basis utilizing the neighboring functions lies in the fact that it minimizes the interactions of basis functions in frequency space [18].

4 The Bernstein-spectral Petrov-Galerkin method

In this section, a Petrov-Galerkin method based on the modal basis functions introduced in Theorem 1 is presented for the time-fractional differential equation

$$\partial_t^\alpha u(x, t) = \sum_{r=0}^n b_r(x, t) \partial_x^r u(x, t) + s(x, t), \quad (x, t) \in \Omega \times (0, T], \quad (4.1)$$

with $\Omega = (0, 1)$, $0 < \alpha \leq 1$, the source term s , the initial condition $u(x, 0) = g(x)$ and n boundary conditions (3.1) in which $v(\cdot) := u(\cdot, t)$. b_i 's are given functions and $\partial_t^\alpha u$ is the Caputo derivative defined by (1.1). Equation (4.1) includes some important problems in science and engineering like the fractional advection-dispersion, the anomalous diffusion, etc. [8]

Let $\tau = \frac{T}{M}$ be the time step length, $t_k = k\tau$ and $u^k(x) := u(x, t_k)$, $0 \leq k \leq M$. The Caputo derivative may be discretized at $t = t_{k+1}$, $k \geq 0$ by the well-known L1 approximation [4]

$$\partial_t^\alpha u(x, t_{k+1}) = \mu_\tau^\alpha \sum_{j=0}^k a_{k,j}^\alpha (u(x, t_{j+1}) - u(x, t_j)) + r_\tau^{k+1}, \quad (4.2)$$

where $\mu_\tau^\alpha = \frac{1}{\tau^\alpha \Gamma(2-\alpha)}$, $a_{k,j}^\alpha = (k+1-j)^{1-\alpha} - (k-j)^{1-\alpha}$ and $|r_\tau^{k+1}| \leq \tilde{c}_u \tau^{2-\alpha}$ in which \tilde{c}_u depends only on u [4]. Using (4.2) in (4.1), we get

$$\mu_\tau^\alpha u^{k+1}(x) - \sum_{r=0}^n b_r^{k+1}(x) \partial_x^r u^{k+1}(x) = f^{k+1}(x), \quad (4.3)$$

where $f^{k+1} = \mu_\tau^\alpha \left(u^k - \sum_{j=0}^{k-1} a_{k,j}^\alpha (u^{j+1} - u^j) \right) + S^{k+1}$.

So at each time step, we need to solve the higher-order differential equation (4.3).

We consider the following Bernstein Petrov-Galerkin formulation for (4.3):

Find $u_N \in V_N^{0,n}$ such that

$$\mu_\tau^\alpha (u_N, v_N) - \sum_{r=0}^n (b_r^{k+1} \partial_x^r u_N, v_N) = (f^{k+1}, v_N), \quad \forall v_N \in W_N^{0,n}. \quad (4.4)$$

Using repeated integration by parts along with conditions (3.2)-(3.1), we can rewrite (4.4) as

$$\mu_\tau^\alpha (u_N, v_N) - \sum_{r=0}^n (-1)^{\lfloor \frac{r+1}{2} \rfloor} (\partial_x^{\lfloor \frac{r}{2} \rfloor} u_N, \partial_x^{\lfloor \frac{r+1}{2} \rfloor} (b_r^{k+1} v_N)) = (f^{k+1}, v_N). \quad (4.5)$$

We expand the approximate solution of (4.3) in terms of the basis functions (3.3) of $V_N^{0,n}$, i.e.,

$$u^{k+1}(x) = \sum_{j=\lfloor \frac{n}{2} \rfloor}^{N-\lfloor \frac{n+1}{2} \rfloor} c_j^{k+1} \phi_j(x). \quad (4.6)$$

Choosing the modal functions introduced in Theorem 1 as the test functions, the (4.5) is written equivalently as

$$\mathbf{A} \mathbf{c}^{k+1} = \mathbf{f}^{k+1}, \quad (4.7)$$

where $\mathbf{f}^{k+1} = [f_i^{k+1} : 0 \leq i \leq N-n]$ and the \mathbf{A} is given by

$$\mathbf{A} = \mu_\tau^\alpha \mathbf{Q} - \sum_{r=0}^n (-1)^{\lfloor \frac{r+1}{2} \rfloor} \mathbf{R}_r, \quad (4.8)$$

where

$$\mathbf{Q} = [(\phi_j, \psi_i)], \quad \mathbf{R}_r = [(\partial_x^{\lfloor \frac{r}{2} \rfloor} \phi_j, \partial_x^{\lfloor \frac{r+1}{2} \rfloor} (\partial_r^{k+1} \psi_i))], \quad 0 \leq r \leq n, \quad (4.9)$$

for $0 \leq i \leq N-n$, $\lfloor \frac{n}{2} \rfloor \leq j \leq N - \lfloor \frac{n+1}{2} \rfloor$. The matrices are $(N-n+1) \times (N-n+1)$. The integrals of \mathbf{f}^{k+1} may be approximated by a Gauss-quadrature rule.

Note that $u^0 = g$ is given by the initial condition and u^{k+1} , $k \geq 0$, are obtained from (4.6) through solving (4.7). By the three-term relation (2.7), (3.4) and the biorthogonality system (2.9), it is found that the matrices in (4.9) (so the coefficient matrix \mathbf{A}) are banded for the problems with constant coefficients.

5 Numerical experiments

Here, we provide some numerical examples to illustrate the accuracy and efficiency of the proposed method.

The errors are measured using the discrete L^∞ as

$$L^\infty := \max_{x \in \Omega} |u(x, T) - u_N^M(x)| \approx \max_{0 \leq i, j \leq \mathcal{N}} |u(x_i, T) - u_N^M(x_i)|,$$

where u is the exact solution of the problem, u_N^M is the approximation solution at $T = t_M = 1$, $x_i = \frac{i}{\mathcal{N}}$ and $\mathcal{N} = 20$.

Example 1. [4] Consider the following time fractional advection-dispersion equation

$$\begin{aligned} \partial_t^\alpha u(x, t) &= \kappa_1 \partial_x^2 u(x, t) - \kappa_2 \partial_x u(x, t) + s(x, t), \quad x \in (0, 1), \\ u(x, 0) &= g(x), \quad u(0, t) = u(1, t) = 0, \end{aligned}$$

where κ_1 and κ_2 are the advection and dispersion coefficients, respectively, and $0 < \alpha \leq 1$. In this problem $n = 2$, so we choose (3.7) as the test functions. The L^∞ errors for the method are reported at $t = 1$ in Table 1 for the case $\kappa_1 = \kappa_2 = 1$ with exact solution $u = \sin(2\pi x) \exp(-t)$ and $\tau = 0.01$.

Example 2. Consider the following equation

$$\begin{aligned} \partial_t^\alpha u(x, t) &= \partial_x u(x, t) + \partial_x^3 u(x, t) - \partial_x^5 u(x, t) + s(x, t), \quad x \in (0, 1), \\ u(x, 0) &= g(x), \quad u(0, t) = u(1, t) = u_x(0, t) = u_x(1, t) = u_{xx}(1, t) = 0, \end{aligned}$$

with the exact solution $u = (1-x) \sin^2(\pi x) \exp(-t)$. Table 2 provides the L^∞ errors of the method with $\tau = 0.01$ for some fractional orders. The spectral accuracy of the method is shown in Figure 5.1 and compared with fixed rates $O(h^r)$, $r = 4, 6$.

N	$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$	
	L^∞	rate	L^∞	rate	L^∞	rate
2	4.31E-01		4.34E-01		4.37E-01	
4	5.94E-02	2.86	5.97E-02	2.86	6.01E-02	2.86
6	3.74E-03	6.82	3.74E-03	6.83	3.74E-03	6.85
8	1.34E-04	11.56	1.34E-04	11.58	1.38E-04	11.48

Table 1: The L^∞ error and the spatial rate of convergence for Example 1.

N	$\alpha = 0.25$		$\alpha = 0.5$		$\alpha = 0.75$	
	L^∞	rate	L^∞	rate	L^∞	rate
6	1.05E-02		1.05E-02		1.05E-02	
8	1.47E-03	6.85	1.47E-03	6.85	1.47E-03	6.85
10	4.55E-05	15.56	4.55E-05	15.56	4.56E-05	15.55
12	7.86E-07	22.26	7.79E-07	22.31	7.49E-07	22.54

Table 2: The L^∞ error and the spatial rate of convergence for Example 2.

6 Conclusion

In this paper, utilizing DBPs, a compact formula for the modal basis functions to solve higher-order BVPs was presented. Using these modal functions, a Bernstein-spectral Petrov-Galerkin method was established for a class of time-fractional PDEs. Some numerical examples have been provided to show the efficiency and spectral accuracy of the method. The proposed method can be implemented for various time fractional PDEs on bounded spatial domains.

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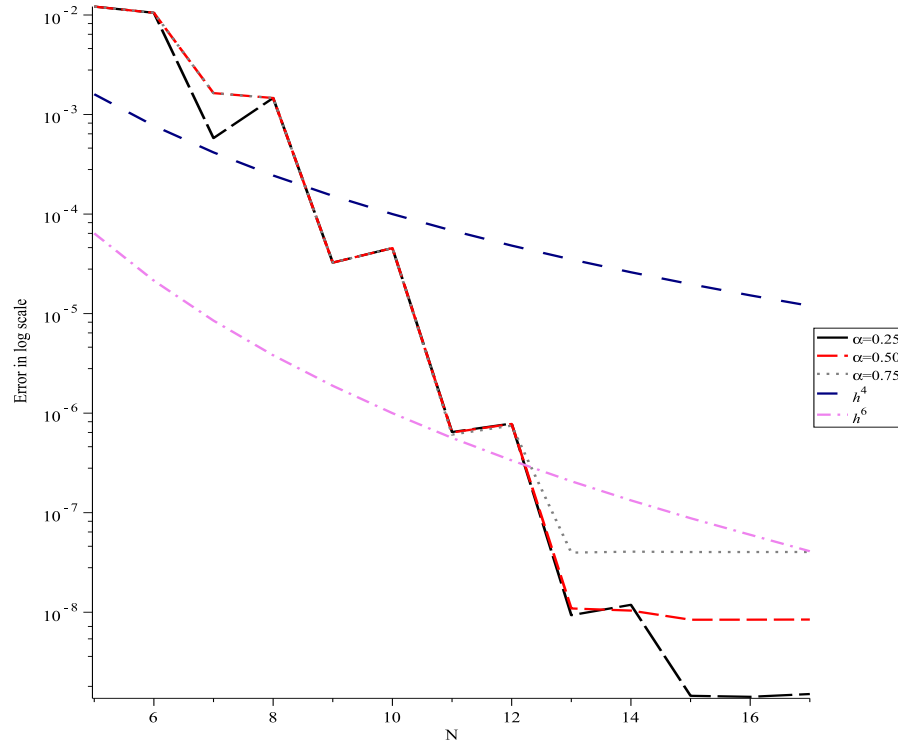


Fig. 5.1: Convergence of the scheme (4.8) and comparison with the fixed polynomial rates $O(h^4)$ and $O(h^6)$.

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